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LETTER TO THE EDITOR

Fractal dimensions of Potts clusters

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Abstract. Potts clusters are connected sets of nearest neighbour sites for which the Potts variable is in the same state. At criticality there exists a fractal cluster. Using arguments from the renormalization group, conformal invariance and numerical simulations, we determine the bulk, surface, hull and red bond dimension of these clusters as a function of the number of Potts states.

The study of fractal growth phenomena has led to a new interest in geometrical properties of spin systems. Geometrical concepts are nowadays regularly used in the study of lattice spin models. We can think, e.g., of the 'droplet' picture introduced for low-dimensional spin glasses [1], or the effective Monte Carlo methods which are based on the use of suitably defined clusters [2].

In the present letter we present results for the fractal properties of clusters in the two-dimensional Potts model [3] at criticality. By *cluster* we mean sets of nearest neighbour sites for which the spins are in the same state. The work presented here extends earlier results for the particular case of the Ising model [4, 5]. The clusters studied here have to be distinguished from the so-called *droplets*, which will be defined below, and whose fractal properties have been determined exactly recently [6, 7].

In the q-state Potts model, one has at each site of a regular lattice a spin variable σ_i which can be in q different states, $\sigma_i = 1, \ldots, q$. The spin interaction is given by the reduced Hamiltonian

$$-\beta H_P = K \sum_{\langle i,j \rangle} (\delta_{\sigma_i \sigma_j} - 1) + h \sum_i (\delta_{\sigma_i 1} - 1). \tag{1}$$

In $d = 2$ and for $h = 0$, this model has a phase transition at some $K_c(q)$ which for $q \leqslant q_c = 4$ is of second order [8]. For $q > q_c$, the transition is first order. The Hamiltonian (1) defines a problem of correlated site percolation for the above defined clusters. Indeed for $K \to \infty$, $h = 0^+$, one expects that all sites will have $\sigma = 1$. Upon increasing the temperature the probability P that a given site of the lattice belongs to the 'infinite' (i.e. lattice spanning) cluster of spins in state 1 will decrease. It was shown rigorously by Coniglio and Peruggi [9] that the percolation threshold for this correlated site percolation problem precisely coincides with $K_c(q)$. For $K \to K_c(q)$ from above, we therefore expect:

$$P \simeq A(K - K_c(q))^{\beta_{per}}. \tag{2}$$

The critical exponent β_{per} has no *a priori* reason (and indeed turns out not) to be equal to the thermodynamic β exponent of the Potts model. Now, given (2), finite size scaling [10] implies that at $K_c(q)$ in a finite system (e.g. a circle of radius R) we have;

$$P(k_c) \sim R^{-\beta_{per}/\nu_{per}}. \tag{3}$$

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This result implies that the 'infinite' cluster at K_c is a fractal of fractal dimension [11] $D = d - \beta_{per}/\nu_{per}$. Besides this bulk fractal dimension we will also be interested in the fractal dimension D_R of red bonds [12] and the fractal dimension D_H of the hull [12] of the infinite cluster (at K_c). Finally, we will also determine the surface fractal dimension D_S [13] which is only defined for semi-infinite systems; it gives the dimension of sites which are both at the surface and in the 'infinite' cluster. The main result of this paper is the determination of these four fractal dimensions for $1 \leqslant q \leqslant 4$; see equations (7) and (8) and (10) and (11). In table 1 we give results for some integer q-values.

Table 1. Bulk (D), surface (D_S), hull (D_H) and red-bond (D_R) fractal dimensions of Potts clusters for some integer q values.

q	D	D_S	D_H	D_R
1	2	1	4/3	−5/12
2	187/96	5/6	11/8	−5/8
3	153/80	3/5	17/12	−17/60
4	15/8	0	3/2	0

We now shortly outline the way in which these results were obtained. A more detailed account will be given elsewhere. First, we have to extend our correlated site percolation problem to a correlated site-bond (CSB) problem in which we place, at random, bonds (with probability p) between the sites of a Potts cluster.

In order to study the properties of the CSB clusters we need a percolative generating function [14]. This generating function can be defined in terms of the average number (per site) of clusters of s sites, $n_s(K, h, p)$;

$$f(K, h, p, H) = \sum_s n_s(K, h, p) \exp(-sH) \tag{4}$$

where H is a 'ghost' magnetic field [14]. A key result for f was determined by Coniglio and Peruggi [9]. They introduced a 'Potts diluted'-Potts model (PdP) in which one couples the original Potts variables σ to other s-state Potts variables μ through the Hamiltonian:

$$-\beta H_{PdP} = -\beta H_P + J \sum_{\langle i,j \rangle} (\delta_{\mu_i \mu_j} - 1)\delta_{\sigma_i 1}\delta_{\sigma_j 1} + H \sum_i (\delta_{\mu_i 1} - 1)\delta_{\sigma_i 1}. \tag{5}$$

They could then show the following relations; $f = dF/ds|_{s=1}$ if $p = 1 - \exp(-J)$ (F is the free energy of the Hamiltonian (5)). For the particular choice $J = K$ the CSB clusters are called droplets and their percolative properties can be completely related to thermodynamic properties of the Potts model (1) [9]. Because the Potts model can be mapped onto the Coulomb gas [15], exact results for fractal dimensions of droplets can be obtained by that technique [6, 7].

The clusters which interest us here are those at $p = 1$. We will determine some of their properties by combining arguments from conformal invariance, the renormalization group (RG), and numerical calculations. A first point to remark is that in the limit $s = 1$ the free energy F will become independent of the variable J ($H = 0$). This free energy in turn uniquely determines the so-called central charge c, which in the theory of conformal invariance at critical points is a crucial quantity [16, 17]. This central charge, which is only defined at scale invariant points, i.e. RG fixed points, is

thus the same at all fixed points which lie on the same line of constant K and h in (K, h, p) space. At its critical point $K = K_c(q)$, $h = 0$, the central charge of the Potts model is given by [18] $c = 1 - 6/m(m+1)$ where m is related to q by $\sqrt{q} = 2\cos(\pi/(m+1))$. In the plane $p = 0$, we have for the rest only trivial fixed points where $c = 0$. Zamolodchikov's theorem [19] (if applicable to these non-unitary theories which we study here), which states that the central charge cannot increase along a RG trajectory, now implies that for $q > 1$ (i.e. $c > 0$) all fixed points on the line $K = K_c(q)$ $h = 0$ must be repulsive in directions perpendicular to the line. What about the RG flow along the line? The point at $p = 0$ is the critical Potts model and can have only two relevant eigenvalues (associated with $|K - K_c|$ and h), and thus the eigenvalue in the J direction must be irrelevant. There must be two more fixed points along the line $K = K_c(q)$, $h = 0$; one at $p = 1 - \exp(-K_c)$ describing critical droplets, another one at still higher p values which coincides with or attracts the critical Potts droplets at $p = 1$. The simplest picture consistent with all these requirements is that the droplet fixed point is fully repulsive and separates the low-J from the high-J region. The complete RG flow is shown in figure 1. It is satisfying that this picture, which we have derived here on quite general grounds, is precisely the one which was found in a Migdal–Kadanoff RG study of Hamiltonian (5) [9]. Secondly, when $q = 1$ it can be seen that f will become independent of K and h that the whole CSB percolation problem reduces to (uncorrelated) random bond percolation. In that limit, the RG flow is of course precisely that which we have derived above for the critical line $K = K_c(q)$, $h = 0$. Our clusters then coincide with the completely trivial fully occupied lattice of bonds at $p = 1$ (so evidently $D = 2$ for $q = 1$), while the droplets are the usual percolation 'clusters' at the percolation threshold. For $q > 1$, the clusters also become non-trivial and their fractal properties are determined by the scaling properties near the tricritical fixed point '1' in figure 1. Conformal invariance is once again very helpful here. Indeed, for a given value of the central charge, conformal invariance restricts the possible values of RG eigenvalues y. For a given m, y can only take on the values $y_{p,q}$ where:

$$y_{p,q} = 2 - [((m+1)p - mq)^2 - 1)]/2m(m+1). \tag{6}$$

Here p and q are restricted to be integers (or half integers [17, 20]. For example, the

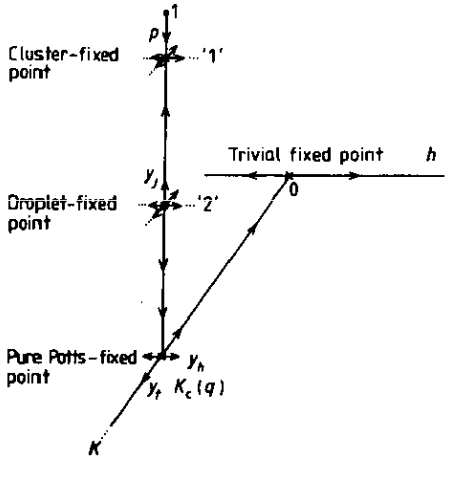

Figure 1. Renormalization group flow for the 'Potts diluted'-Potts model (equation (5)) for $H = 0$, $p = 1 - \exp(-J)$. Fixed points are denoted by crosses.

thermal eigenvalue y_t of the Potts model is given by $y_{2,1}$ while the magnetic eigenvalue y_h equals $y_{(m+1)/2,(m-1)/2}$ [18]. These eigenvalues also describe the flow away from the critical line at non-zero p. The third relevant eigenvalue y_J at the droplet fixed point was determined by Coniglio [7] to be $y_{m,m+1}=(4m+1)/2m(m+1)$. This has a very important consequence, not pointed out in [7]. When $q\to 4$ ($m\to\infty$), we have $y_J\to 0$. Such a marginal eigenvalue is always associated with the coalescence of two fixed points. Clearly, the droplet fixed point cannot coalesce with the pure Potts fixed point, so we conclude that as q goes to 4 the droplet and cluster fixed point must come together. Remember that $q=q_c=4$ also is the critical value above which the Potts transition becomes first order (it is interesting to note that in the approximate RG of [9] such a coalescence is indeed observed at the value $q=16$). Figure 2 shows the flow along the critical line, as a function of q. Notice the similarity between this picture and the RG flow of the Potts model including annealed vacancies [21]. Our results imply that at $q=4$ clusters and droplets become identical. Because of the eigenvalue $y_J=0$, we expect logarithmic corrections in geometric quantities at $q=4$ [22]. We thus recover in the cluster-droplet properties of the Potts model a behaviour which is well known for its thermal properties [21].

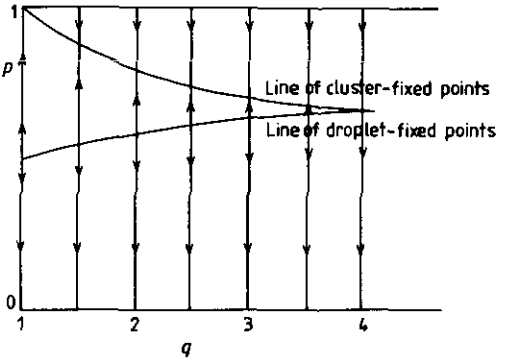

Figure 2. The critical line ($K=K_c(q)$, $h=0$) as a function of q shows the line of droplet fixed points (known exactly and shown here for the square lattice case) and the line of cluster fixed points (whose precise location is unknown) which coalesce at $q=4$.

The fractal dimensions of Ising clusters (i.e. the special case $q=2$) were recently determined from a mapping onto the tricritical one-state Potts model [4, 5]. Such a mapping is not possible for general q. Yet, we are still able to determine the fractal dimensions of Potts clusters combining the results presented so far.

Indeed, we know D and D_S for $q=1$ (relation to bond percolation), $q=2$ (equivalence with tricritical one-state Potts model) and for $q=4$ (where clusters are droplets). Combining this with the knowledge of the central charge c and the limited sets of values for y given by (6), uniquely determines D and D_S as a function of q. The result is

$$D=y_{m/2,m/2}=(15m^2+16m+4)/8m(m+1) \tag{7}$$

and†

$$D_S=y'_{3,5}=(4m-2)/m(m+1). \tag{8}$$

† For surface critical exponents, possible values are given by $y'=y_{p,q}/2$ where $y_{p,q}$ is given by (6).

We have checked our predictions for $q=3$ ($m=5$) numerically. The technique we used was applied earlier to the case of the Ising clusters [5]. For systems of finite size R (in our case squares of side R, for $8 \leqslant R \leqslant 36$) we calculated with a Monte Carlo technique the percolative susceptibility χ defined as

$$\chi(K, h) = \frac{\partial^2}{\partial H^2} f(K, h, p=1, H)|_{H=0} = \sum_s s^2 n_s(K, h, p=1) \tag{9}$$

which is a finite system at criticality should diverge as $R^{**}(-2+2D)$ [10]. From our data we find $D=1.91 \pm 0.01$, in excellent agreement with the prediction from (7) $D=153/80=1.9125$. A similar agreement between numerical and exact result was found for the case $q=2$ [5]. It is interesting to remark that recently estimates for D for $q=2, 3$ and 4 were calculated using the fixed scale transformation [23], a technique introduced to calculate analytically fractal dimensions of spin and growth models [24]. The values found for D by that method are very close to the exact ones determined here.

For the red bond and hull fractal dimension we can only use results for $q=2$ [5] and $q=4$ [7]. Yet, knowledge of c and equation (6) lead almost uniquely to the conjectures:

$$D_H = y_{m+1,m+1} = (3m+2)/2(m+1) \tag{10}$$

and

$$D_R = y_{m+2,m+1} = (-4m+3)/2m(m+1). \tag{11}$$

The red bond dimension D_R turns out to be negative for $1 \leqslant q < 4$. This is consistent with our RG picture (figure 2) as it can be shown [7] that D_R equals the RG eigenvalue in the p (or J) direction at the cluster fixed point, which clearly is negative. A negative fractal dimension can be interpreted in the following way: bonds which when considering only a small part of the fractal seem to be red bonds, turn out not to be red bonds when the fractal is considered on larger and larger length scales. Therefore, on big enough scales, Potts clusters are 'fat' fractals containing no red bonds. Notice finally that for $q=1$, $D_R=-\frac{5}{12}$ gives a new exact result for bond percolation; it is the irrelevant exponent describing the RG flow near $p=1$. The hull fractal dimension for $q=1$ becomes $\frac{4}{3}$, a result which is not immediately understandable because the usual hull dimension loses its meaning for a cluster occupying the whole lattice. The correct interpretation for $D=\frac{4}{3}$ follows if we extend recent work of Seno *et al* [25] for Ising clusters to the Potts case. Then, it can be shown (details will appear elsewhere) that the hull dimension is also equal to the fractal dimension of a self-avoiding walk (SAW) which is restricted to be on a cluster of fixed spin state. This definition keeps meaning for $q=1$, and in that case the dimension of the SAW is just that of one on a pure lattice which indeed equals $\frac{4}{3}$ [26].

In summary, we have determined several fractal dimensions for Potts clusters as a function of q. These results extend similar results for Ising clusters and Potts droplets. One interesting dimension which so far remains undetermined, even in the simplest case of uncorrelated percolation, is the backbone dimension [12].

Several further interesting results can be obtained for q in the neighbourhood of 4. Using selection rules from conformal invariance the precise form of logarithmic corrections can be found [27]. Then, using well established techniques [28] one can e.g. deduce that the percolative susceptibility (equation (9)) for the four-state Potts model diverges as

$$\chi(K, h=0) \sim t^{-7/6} |\ln t|^{-1/8} \tag{12}$$

for $t \sim |K - K_c(4)| \to 0$. Details of this calculation, and other related results, will be given elsewhere.

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